An application of the Riemann zeta function

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April 5, 2010

PME Journal Spring 2009 Problem 1196

- The Pi Mu Epsilon Journal Problem Section for Spring 2009

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- Problem 1196:
- Let $\mathbb{Q}^* = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, a \neq 0, b > 0, \text{ and } \gcd(a, b) = 1\}$. In other words, $\mathbb{Q}^*$ is the set of all nonzero rational numbers written in lowest terms. Find, with proof, the value of

$$\sum_{\frac{a}{b} \in \mathbb{Q}^*} \frac{1}{(ab)^2}.$$

Definitions and Theorems I

- Riemann zeta function

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- Multiplicative function

An arithmetic function, $f(n)$, is multiplicative if, given $a, b \in \mathbb{Z}^+$ and $\gcd(a, b) = 1$, $f(a)f(b) = f(ab)$.

Definitions and Theorems II

- Euler product

  If $f(n)$ is a real or complex-valued multiplicative function such that $\sum_{n=1}^{\infty} |f(n)| < \infty$, then

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f\left(p^2\right) + f\left(p^3\right) + \cdots\right).$$

Definitions and Theorems II

- Euler product

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$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f\left(p^2\right) + f\left(p^3\right) + \cdots\right).$$

- Euler's identity

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

Redefining the Problem I

So

$$\sum_{\frac{a}{b}\in\mathbb{Q}^*} \frac{1}{(ab)^2}.$$

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Removing the negative values of a yields $\mathbb{Q}^+ = \{\frac{a}{b} \mid a, b \in \mathbb{Z}^+$ and $\gcd(a, b) = 1\}$.

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Removing the negative values of a yields $\mathbb{Q}^+ = \{\frac{a}{b} \mid a, b \in \mathbb{Z}^+$ and $\gcd(a, b) = 1\}$. Then

$$\sum_{\frac{a}{b}\in\mathbb{Q}^*} \frac{1}{(ab)^2} = 2\sum_{\frac{a}{b}\in\mathbb{Q}^+} \frac{1}{(ab)^2}.$$

Redefining the Problem II

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$$ab = n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$$

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$$a = 1 \ ,b = 60 \quad a = 3 \ ,b = 20 \quad a = 4 \ ,b = 15 \quad a = 5 \ ,b = 12$$

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Suppose there exists $k, \ell \in \mathbb{Z}^+$ such that $\gcd(k, \ell) = 1$. Then,

$$f(k)\, f(\ell) = \frac{2^{\omega(k)}}{k^s}\ \frac{2^{\omega(\ell)}}{\ell^s} = \frac{2^{\omega(k)+\omega(\ell)}}{k^s \ell^s} = \frac{2^{\omega(k\ell)}}{(k\ell)^s} = f(k\ell)$$

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So

$$\begin{aligned} n &> 4^{r-2} \\ \log_2 n &> \log_2 \left(2^{2(r-2)}\right) = 2\left(r-2\right) \\ r &< \frac{1}{2}\log_2 n + 2. \end{aligned}$$

Show $\sum f(n)$ converges absolutely II

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$$2^{\omega(n)} = 2^r < \left(2^{\log_2 n}\right)^{\frac{1}{2}} 2^2$$

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$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} < \sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^s} = \sum_{n=1}^{\infty} \frac{4}{n^{s-\frac{1}{2}}}.$$

A few manipulations I

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f\left(p^2\right) + f\left(p^3\right) + \cdots\right)$$

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$$= \prod_p \left(1 + \frac{2}{p^s}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)\right)$$

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$$= \prod_p \left(1 + \frac{2}{p^s} \left(\frac{1}{1 - p^{-s}}\right)\right)$$

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$$= \prod_p \left(1 - p^{-s}\right)^{-1} \left(\left(1 - p^{-s}\right) + \frac{2}{p^s}\right)$$

A few manipulations II

$$
\begin{aligned}
&= \prod_p \left(1 + \frac{2}{p^s}\left(\frac{1}{1-p^{-s}}\right)\right) \\
&= \prod_p \left(1 + \frac{2}{p^s}\left(1-p^{-s}\right)^{-1}\right) \\
&= \prod_p \left(1-p^{-s}\right)^{-1}\left(\left(1-p^{-s}\right) + \frac{2}{p^s}\right) \\
&= \prod_p \left(1-p^{-s}\right)^{-1}\left(1+p^{-s}\right)
\end{aligned}
$$

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\begin{aligned}
&= \prod_p \left(1 + \frac{2}{p^s} \left(\frac{1}{1 - p^{-s}}\right)\right) \\
&= \prod_p \left(1 + \frac{2}{p^s} \left(1 - p^{-s}\right)^{-1}\right) \\
&= \prod_p \left(1 - p^{-s}\right)^{-1} \left(\left(1 - p^{-s}\right) + \frac{2}{p^s}\right) \\
&= \prod_p \left(1 - p^{-s}\right)^{-1} \left(1 + p^{-s}\right) \\
&= \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 + p^{-s}\right)
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$$

A few manipulations III

$$= \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 + p^{-s}\right)$$

A few manipulations III

$$= \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 + p^{-s}\right)$$

$$= \zeta\left(s\right) \prod_p \left(1 + p^{-s}\right)$$

A few manipulations III

$$= \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 + p^{-s}\right)$$

$$= \zeta(s) \prod_p \left(1 + p^{-s}\right)$$

$$= \zeta(s) \prod_p \left(1 + p^{-s}\right) \frac{1 - p^{-s}}{1 - p^{-s}}$$

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$$= \zeta(s) \prod_p \frac{1 - p^{-2s}}{1 - p^{-s}}$$

A few manipulations III

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\begin{aligned}
&= \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 + p^{-s}\right) \\
&= \zeta(s) \prod_p \left(1 + p^{-s}\right) \\
&= \zeta(s) \prod_p \left(1 + p^{-s}\right) \frac{1 - p^{-s}}{1 - p^{-s}} \\
&= \zeta(s) \prod_p \frac{1 - p^{-2s}}{1 - p^{-s}} \\
&= \zeta(s) \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 - p^{-2s}\right)
\end{aligned}
$$

A few manipulations IV

$$= \zeta(s) \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 - p^{-2s}\right)$$

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$$
\begin{aligned}
&= \zeta(s) \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 - p^{-2s}\right) \\
&= \zeta^2(s) \prod_p \left(1 - p^{-2s}\right)
\end{aligned}
$$

A few manipulations IV

$$= \zeta(s) \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 - p^{-2s}\right)$$

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$$= \zeta^2(s) \left(\prod_p \left(1 - p^{-2s}\right)^{-1} \right)^{-1}$$

A few manipulations IV

$$
\begin{aligned}
&= \zeta(s) \prod_p \left(1 - p^{-s}\right)^{-1} \prod_p \left(1 - p^{-2s}\right) \\
&= \zeta^2(s) \prod_p \left(1 - p^{-2s}\right) \\
&= \zeta^2(s) \left(\prod_p \left(1 - p^{-2s}\right)^{-1} \right)^{-1} \\
&= \zeta^2(s)\, \zeta^{-1}(2s)
\end{aligned}
$$

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&= \zeta^2(s)\, \zeta^{-1}(2s) \\
&= \frac{\zeta^2(s)}{\zeta(2s)}
\end{aligned}
$$

Back to the problem

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$$

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$$2\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^2} = 2\frac{\zeta^2(2)}{\zeta(4)}$$

A solution at last

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

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$$2\frac{\zeta^2(2)}{\zeta(4)} = 2\frac{\left(\frac{\pi^2}{6}\right)^2}{\left(\frac{\pi^4}{90}\right)} = 2\left(\frac{\pi^4}{36}\right)\left(\frac{90}{\pi^4}\right) = 5$$

Sum it up

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Sum it up

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Sum it up

$$\sum_{\frac{a}{b}\in\mathbb{Q}^*} \frac{1}{(ab)^2} = 2\sum_{\frac{a}{b}\in\mathbb{Q}^+} \frac{1}{(ab)^2} = 2\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^2} = 2\frac{\zeta^2(2)}{\zeta(4)} = 5$$

References

Ivić, Aleksandar, *The Riemann Zeta-Function,* John Wiley & Sons, Inc., 1985.

Appendix

Formula for calculating even values of $\zeta(s)$

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!}$$

Generating function for the Bernoulli numbers

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!} \quad (|z| < 2\pi)$$

$$B_2 = \frac{1}{6} \text{ and } B_4 = -\frac{1}{30}$$

$$\zeta(2) = \frac{(-1)^{1+1}(2\pi)^{2(1)} B_{2(1)}}{2(2(1))!} = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{(-1)^{2+1}(2\pi)^{2(2)} B_{2(2)}}{2(2(2))!} = \frac{\pi^4}{90}$$